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Some Properties for Convolutions of Generalized Hypergeometric Functions

Shigeyoshi Owa, Ji A Kim and Nak Eun Cho

Abstract

With the convolution products of generalized hypergeometric functions ${}_pF_q(z)$ and analytic functions $f(z)$ in the open unit disk, the operator $I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_p}(f)$ is introduced. The object of the present paper is to derive some interesting properties of operator $I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_p}(f)$ associated with some classes of univalent functions.

1. Introduction

Let \mathcal{A} denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Denote by \mathcal{S} the class of all functions in \mathcal{A} which are univalent in U .

A function $f(z) \in \mathcal{A}$ is said to be in the class $R^t(A, B)$ if

$$(1.2) \quad \left| \frac{f'(z) - 1}{t(A - B) - B(f'(z) - 1)} \right| < 1,$$

where A and B are arbitrary fixed numbers with $-1 \leq B < A \leq 1$ and $t \in \mathbb{C} \setminus \{0\}$ (\mathbb{C} is the set of all complex numbers). Clearly, a function $f(z)$ belongs to $R^t(A, B)$ if and only if there exists a function $w(z)$ regular in U satisfying $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$) such that

$$(1.3) \quad 1 + \frac{1}{t}(f'(z) - 1) = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in U).$$

The class $R^t(A, B)$ was introduced by Dixit and Pal [4], recently.

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By giving specific values t , A and B in (1.2), we obtain the following subclasses studied by various researchers in earlier works :

(i) For $t = e^{-i\eta} \cos \eta$ ($|\eta| < \frac{\pi}{2}$), $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $B = -1$, we obtain the class of functions f satisfying the condition

$$(1.4) \quad \left| \frac{e^{i\eta}(f'(z) - 1)}{2(1 - \alpha) \cos \eta + e^{i\eta}(f'(z) - 1)} \right| < 1 \quad (z \in U).$$

In this case, the class $R^t(A, B)$ is equivalent to the class $R_\eta(\alpha)$ which is studied by Ponnusamy and Rønning [11]. Here $R_\eta(\alpha)$ is the class of functions $f(z) \in \mathcal{A}$ satisfying the condition

$$\operatorname{Re}(e^{i\eta}(f'(z) - \alpha)) > 0 \quad (|\eta| < \frac{\pi}{2}, 0 \leq \alpha < 1, z \in U).$$

(ii) For $t = e^{-i\eta} \cos \eta$ ($|\eta| < \frac{\pi}{2}$), we obtain the class of functions $f(z) \in \mathcal{A}$ satisfying the condition

$$\left| \frac{e^{i\eta}(f'(z) - 1)}{Be^{i\eta}f'(z) - (A \cos \eta + iB \sin \eta)} \right| < 1 \quad (z \in U),$$

which was studied by Dashrath [3].

(iii) For $t = 1$, $A = \beta$ and $B = -\beta$ ($0 < \beta \leq 1$), we obtain the class of functions $f(z)$ satisfying the condition

$$\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta \quad (z \in U),$$

which was studied by Padmanabhan [10] and Caplinger and Cauchy [2].

Let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote the subclasses of \mathcal{S} consisting of starlike and convex functions of order α ($0 \leq \alpha < 1$) in U , respectively. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) \equiv \mathcal{S}^*$, $\mathcal{C}(\alpha) \subset \mathcal{C}(0) \equiv \mathcal{C}$ and $\mathcal{C}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}$. For $\lambda > 0$, define the classes \mathcal{S}_λ^* and \mathcal{C}_λ by

$$\mathcal{S}_\lambda^* = \{f(z) \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} \right| < \lambda, z \in U\}$$

and

$$\mathcal{C}_\lambda = \{f(z) \in \mathcal{A} : zf'(z) \in \mathcal{S}_\lambda^*\},$$

respectively. It is a known fact that a sufficient condition for $f(z) \in \mathcal{A}$ of the form (1.1) to belong to the class \mathcal{S}^* is that $\sum_{n=2}^{\infty} n|a_n| \leq 1$. A simple extension of this result is the following [16] :

$$(1.5) \quad \sum_{n=2}^{\infty} (n + \lambda - 1)|a_n| \leq \lambda \implies f(z) \in \mathcal{S}_\lambda^*.$$

For $\lambda = \frac{1}{2}$, this was previously proved by Schild [18]. Since $f(z) \in \mathcal{C}_\lambda$ if and only if $zf'(z) \in \mathcal{S}_\lambda^*$, we have a corresponding result for \mathcal{C}_λ ,

$$(1.6) \quad \sum_{n=2}^{\infty} n(n+\lambda-1)|a_n| \leq \lambda \implies f(z) \in \mathcal{C}_\lambda.$$

In this paper, we consider the generalized hypergeometric series ${}_pF_q(z)$ defined by

$$(1.7) \quad {}_pF_q(z) \equiv {}_pF_q \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_n}{\prod_{i=1}^q (b_i)_n} \frac{z^n}{(1)_n}$$

where p and q are positive integers and we assume that the variable z , the numerator parameters a_1, a_2, \dots, a_p and the denominator parameters b_1, b_2, \dots, b_q take on complex values, provided that $b_i \neq 0, -1, -2, \dots$; $i = 1, 2, \dots, q$. Here $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1 & \text{if } n = 0 \\ \lambda(\lambda+1) \cdots (\lambda+n-1) & \text{if } n \in N = \{1, 2, \dots\}. \end{cases}$$

For any complex number λ , we also use the ascending factorial notation

$$(1.8) \quad (\lambda)_n = \lambda(\lambda+1)_{n-1}$$

for $n \geq 1$ and $(\lambda)_0 = 1$ for $\lambda \neq 0$. If λ is neither zero nor a negative integer, then using the definition of the Gamma function, we can write

$$(1.9) \quad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}.$$

Furthermore, if we set

$$\omega = \sum_{i=1}^q b_i - \sum_{j=1}^p a_j$$

it is known that the series ${}_pF_q(z)$, with $p = q + 1$, is

- (i) absolutely convergent for $|z| = 1$ if $\operatorname{Re} \omega > 0$,
 - (ii) conditionally convergent for $|z| = 1$, $z \neq 1$ if $-1 < \operatorname{Re} \omega \leq 0$
- and

- (iii) divergent for $|z| = 1$ if $\operatorname{Re} \omega \leq -1$.

As in the case of the function ${}_2F_1(z)$, we are led to the well-known Gauss summation theorem :

$$(1.10) \quad {}_2F_1 \left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix} ; 1 \right) = \frac{\Gamma(b_1)\Gamma(b_1 - a_1 - a_2)}{\Gamma(b_1 - a_1)\Gamma(b_1 - a_2)}, \quad \operatorname{Re}(b_1 - a_1 - a_2) > 0.$$

We recall that the function ${}_2F_1(z)$ is bounded if $\operatorname{Re}(b_1 - a_1 - a_2) > 0$ and has a pole at $z = 1$ if $\operatorname{Re}(b_1 - a_1 - a_2) \leq 0$ (cf. [1]). Univalence, starlikeness and convexity properties of $z {}_2F_1\left(\begin{smallmatrix} a_1, a_2 \\ b_1 \end{smallmatrix}; z\right)$ have been studied extensively in [12, 15].

For $f(z) \in \mathcal{A}$, we define the operator $I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_p}(f)$ by

$$(1.11) \quad \left[I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_p}(f) \right](z) = z {}_pF_q \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix}; z \right) * f(z),$$

where the symbol " $*$ " denotes the usual Hadamard product or convolution of power series.

2. Properties of the operators with $R^t(A, B)$

Now we introduce several lemmas which are needed for the proof of our main results.

Lemma 2.1 ([8]) *Let $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_1 \in U$, then we can write*

$$z_1 w'(z_1) = m w(z_1),$$

where m is real and $m \geq 1$.

Lemma 2.2 ([4]) *Let a function $f(z)$ of the form (1.1) be in $R^t(A, B)$. Then*

$$|a_n| \leq \frac{(A - B)|t|}{n}.$$

Then result is sharp for the function

$$f(z) = \int_0^z \left(1 + \frac{(A - B)t z^{n-1}}{1 + B z^{n-1}} \right) dz \quad (n \geq 2, z \in U).$$

Lemma 2.3 ([4]) *Let a function $f(z)$ of the form (1.1) be in \mathcal{A} . If*

$$\sum_{n=2}^{\infty} (1 + |B|)n|a_n| \leq (A - B)|t| \quad (-1 \leq B < A \leq 1, t \in C \setminus \{0\})$$

then $f(z) \in R^t(A, B)$. The result is sharp for function

$$f(z) = z + \frac{(A - B)t}{(1 + |B|)n} z^n \quad (n \geq 2, z \in U).$$

Our first result for the operators is contained in

Theorem 2.1 *If $f(z) \in \mathcal{A}$ satisfies*

$$(2.1) \quad \left| \frac{I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_p, 2}(f)}{z} - 1 \right|^{1-\beta} \left| \frac{I_{b_1, b_2, \dots, b_q, 1}^{a_1, a_2, \dots, a_p, 2, 2}(f)}{I_{b_1, b_2, \dots, b_q, 1}^{a_1, a_2, \dots, a_p, 2}(f)} - 1 \right|^\beta < \left(\frac{1}{2}\right)^\beta$$

for some fixed $\beta \geq 0$, then $I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_p}(f)$ is univalent (close-to-convex) in U .

Proof. We note that

$$I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_p}(f) = z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n-1}}{\prod_{i=1}^q (b_i)_{n-1} (1)_{n-1}} a_n z^n$$

in \mathcal{A} . Define $w(z)$ by

$$w(z) = \frac{I_{b_1, b_2, \dots, b_q, 1}^{a_1, a_2, \dots, a_p, 2}(f)}{z} - 1$$

for $z \in U$. Then it follows that $w(z)$ is analytic in U with $w(0) = 0$. By (2.1), it is clear that

$$(2.2) \quad |w(z)|^{1-\beta} \left| \frac{zw'(z)}{1+w(z)} \right|^\beta = |w(z)| \left| \frac{zw'(z)}{w(z)(1+w(z))} \right|^\beta < \left(\frac{1}{2}\right)^\beta.$$

Suppose that there exists a point $z_1 \in U$ such that

$$\max_{|z| \leq |z_1|} |w(z)| = |w(z_1)| = 1.$$

Then we can put

$$\frac{z_1 w'(z_1)}{w(z_1)} = m \geq 1,$$

by Lemma 2.1. Therefore we obtain

$$|w(z_1)| \left| \frac{z_1 w'(z_1)}{w(z_1)(1+w(z_1))} \right|^\beta \geq \left(\frac{m}{2}\right)^\beta \geq \left(\frac{1}{2}\right)^\beta,$$

which contradicts the condition (2.2). This shows that

$$|w(z)| = \left| \frac{I_{b_1, b_2, \dots, b_q, 1}^{a_1, a_2, \dots, a_p, 2}(f)}{z} - 1 \right| < 1,$$

which implies that $\operatorname{Re} \left[I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_p}(f) \right]'(z) > 0$ for $z \in U$. Therefore, by Noshiro-Warschawski Theorem [5], $I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_p}(f)$ is univalent (close-to-convex) in U .

Theorem 2.2 Let a_j ($j = 1, 2, \dots, p$) $\in C \setminus \{0\}$, b_i ($i = 1, 2, \dots, q$) $\in C \setminus \{0\}$, $\text{Re} b_i > 0$ ($i = 1, 2, \dots, q$), and $\sum_{i=1}^q \text{Re} b_i > \sum_{j=1}^p |a_j|$. If $f(z) \in R^t(A, B)$ satisfies

$$(2.3) \quad {}_pF_q \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p| \\ \text{Re} b_1, \text{Re} b_2, \dots, \text{Re} b_q \end{matrix} ; 1 \right) \leq \frac{1}{1 + |B|} + 1,$$

then

$$z {}_pF_q \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} ; z^k \right) * f(z) \in R^t(A, B),$$

where $k \in N$.

Proof. By Lemma 2.3, it suffices to show that

$$(2.4) \quad T_1 := \sum_{n=2}^{\infty} (1 + |B|)(k(n-1) + 1) \left| \frac{\prod_{j=1}^p (a_j)_{n-1}}{\prod_{i=1}^q (b_i)_{n-1} (1)_{n-1}} a_{k(n-1)+1} \right| \leq (A - B)|t|.$$

From Lemma 2.2 and the fact that $|(a)_n| \leq (|a|)_n$ and $(\text{Re} b)_n \leq |(b)_n|$, $\text{Re} b > 0$, we have

$$\begin{aligned} T_1 &\leq \sum_{n=2}^{\infty} (A - B)(1 + |B|)|t| \left\{ \frac{\prod_{j=1}^p (|a_j|)_{n-1}}{\prod_{i=1}^q (\text{Re} b_i)_{n-1} (1)_{n-1}} \right\} \\ &= (A - B)(1 + |B|)|t| \left\{ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (|a_j|)_n}{\prod_{i=1}^q (\text{Re} b_i)_n (1)_n} - 1 \right\} \\ &= (A - B)(1 + |B|)|t| \left\{ {}_pF_q \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p| \\ \text{Re} b_1, \text{Re} b_2, \dots, \text{Re} b_q \end{matrix} ; 1 \right) - 1 \right\} \\ &\leq (A - B)|t| \end{aligned}$$

by (2.3). This completes the proof of Theorem 2.2.

Corollary 2.1 Let a_j ($j = 1, 2, \dots, q+1$) $\in C \setminus \{0\}$, b_i ($i = 1, 2, \dots, q$) $\in C \setminus \{0\}$, $\text{Re} b_m > |a_m| + 1$ ($m = 1, 2, \dots, q-1$), and $\text{Re} b_q > |a_q| + |a_{q+1}|$. If $f(z) \in R^t(A, B)$ satisfies

$$(2.5) \quad \frac{\Gamma(\text{Re} b_q) \Gamma(\text{Re} b_q - |a_q| - |a_{q+1}|)}{\Gamma(\text{Re} b_q - |a_q|) \Gamma(\text{Re} b_q - |a_{q+1}|)} \left(\prod_{m=1}^{q-1} \frac{\text{Re} b_m - 1}{\text{Re} b_m - |a_m| - 1} \right) \leq \frac{1}{1 + |B|} + 1,$$

then

$$z {}_{q+1}F_q \left(\begin{matrix} a_1, a_2, \dots, a_{q+1} \\ b_1, b_2, \dots, b_q \end{matrix} ; z^k \right) * f(z) \in R^t(A, B),$$

where $k \in N$.

Proof. We note that

$$\begin{aligned}
 (2.6) \quad {}_{q+1}F_q \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_{q+1}| \\ \text{Re}b_1, \text{Re}b_2, \dots, \text{Re}b_q \end{matrix} ; 1 \right) &= \sum_{n=0}^{\infty} \frac{(|a_1|)_n \cdots (|a_{q+1}|)_n}{(\text{Re}b_1)_n \cdots (\text{Re}b_q)_n (1)_n} \\
 &= \left(\prod_{m=1}^{q-1} \frac{\Gamma(\text{Re}b_m) \Gamma(\text{Re}b_m - |a_m| - 1)}{\Gamma(\text{Re}b_m - |a_m|) \Gamma(\text{Re}b_m - 1)} \right) \frac{\Gamma(\text{Re}b_q) \Gamma(\text{Re}b_q - |a_q| - |a_{q+1}|)}{\Gamma(\text{Re}b_q - |a_q|) \Gamma(\text{Re}b_q - |a_{q+1}|)} \\
 &= \left(\prod_{m=1}^{q-1} \frac{\text{Re}b_m - 1}{\text{Re}b_m - |a_m| - 1} \right) \frac{\Gamma(\text{Re}b_q) \Gamma(\text{Re}b_q - |a_q| - |a_{q+1}|)}{\Gamma(\text{Re}b_q - |a_q|) \Gamma(\text{Re}b_q - |a_{q+1}|)}.
 \end{aligned}$$

Hence we have

$$\begin{aligned}
 &\sum_{n=2}^{\infty} (1 + |B|)(k(n-1) + 1) \left| \frac{\prod_{j=1}^{q+1} (a_j)_{n-1}}{\prod_{i=1}^q (b_i)_{n-1} (1)_{n-1}} a_{k(n-1)+1} \right| \\
 &\leq (A - B)(1 + |B|)|t| \left\{ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{q+1} (|a_j|)_n}{\prod_{i=1}^q (\text{Re}b_i)_n (1)_n} - 1 \right\} \\
 &= (A - B)(1 + |B|)|t| \left\{ {}_{q+1}F_q \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_{q+1}| \\ \text{Re}b_1, \text{Re}b_2, \dots, \text{Re}b_q \end{matrix} ; 1 \right) - 1 \right\} \\
 &\leq (A - B)|t|,
 \end{aligned}$$

by assumption. This completes our proof.

Theorem 2.3 Let a_j ($j = 1, 2, \dots, p$) $\in C \setminus \{0\}$, b_i ($i = 1, 2, \dots, q$) $\in C \setminus \{0\}$, $\text{Re}b_i > 0$ ($i = 1, 2, \dots, q$), and $\sum_{i=1}^q \text{Re}b_i > \sum_{j=1}^p |a_j|$. If $f(z) \in R^t(A, B)$ satisfies

$$(2.7) \quad {}_{p+2}F_{q+2} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, \lambda + 1, 1 \\ \text{Re}b_1, \text{Re}b_2, \dots, \text{Re}b_q, \lambda, 2 \end{matrix} ; 1 \right) \leq \frac{1}{(A - B)|t|} + 1,$$

then $I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_p}(f) \in S_{\lambda}^*$ where $\lambda > 0$.

Proof. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R^t(A, B)$. Then, by (1.5) it suffices to show that

$$(2.8) \quad T_2 := \sum_{n=2}^{\infty} (n + \lambda - 1) \left| \frac{\prod_{j=1}^p (a_j)_{n-1}}{\prod_{i=1}^q (\text{Re}b_i)_{n-1} (1)_{n-1}} a_n \right| \leq \lambda.$$

From Lemma 2.3, we observe that

$$T_2 \leq \sum_{n=2}^{\infty} (n + \lambda - 1) \frac{(A - B)|t|}{n} \left\{ \frac{\prod_{j=1}^p (|a_j|)_{n-1}}{\prod_{i=1}^q (\text{Re}b_i)_{n-1} (1)_{n-1}} \right\}$$

$$\begin{aligned}
&= (A - B)|t| \sum_{n=1}^{\infty} \frac{\lambda(\lambda+1)_n}{(\lambda)_n} \frac{(1)_n}{(2)_n} \left\{ \frac{\prod_{j=1}^p (|a_j|)_n}{\prod_{i=1}^q (\text{Re}b_i)_n (1)_n} \right\} \\
&= \lambda(A - B)|t| \left\{ {}_{p+2}F_{q+2} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, \lambda+1, 1 \\ \text{Re}b_1, \text{Re}b_2, \dots, \text{Re}b_q, \lambda, 2 \end{matrix} ; 1 \right) - 1 \right\} \\
&\leq \lambda
\end{aligned}$$

by (2.7), which completes the proof of Theorem 2.3.

Corollary 2.2 Let a_j ($j = 1, 2, \dots, q+1$) $\in C \setminus \{0\}$, b_i ($i = 1, 2, \dots, q$) $\in C \setminus \{0\}$, $\text{Re}b_m > |a_m| + 1$ ($m = 1, 2, \dots, q-1$), $|a_q| < 1$ and $\text{Re}b_q - 2 > \lambda > |a_{q+1}| + 1$. If $f(z) \in R^t(A, B)$ satisfies

$$\begin{aligned}
(2.9) \quad & \frac{(\lambda-1)(\text{Re}b_q-1)}{(1-|a_q|)(\lambda-|a_{q+1}|-1)(\text{Re}b_q-\lambda-2)} \\
& \times \left(\prod_{m=1}^{q-1} \frac{\text{Re}b_m-1}{\text{Re}b_m-|a_m|-1} \right) \leq \frac{1}{(A-B)|t|} + 1,
\end{aligned}$$

then $I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_{q+1}}(f) \in S_\lambda^*$ where $\lambda > 0$.

Proof. We note that

$$\begin{aligned}
& {}_{q+3}F_{q+2} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_{q+1}|, \lambda+1, 1 \\ \text{Re}b_1, \text{Re}b_2, \dots, \text{Re}b_q, \lambda, 2 \end{matrix} ; 1 \right) \\
&= \left(\prod_{m=1}^{q-1} \frac{\Gamma(\text{Re}b_m)\Gamma(\text{Re}b_m-|a_m|-1)}{\Gamma(\text{Re}b_m-|a_m|)\Gamma(\text{Re}b_m-1)} \right) \frac{\Gamma(2)\Gamma(1-|a_q|)\Gamma(\lambda)\Gamma(\lambda-|a_{q+1}|-1)}{\Gamma(2-|a_q|)\Gamma(1)\Gamma(\lambda-|a_{q+1}|)\Gamma(\lambda-1)} \\
& \quad \times \frac{\Gamma(\text{Re}b_q)\Gamma(\text{Re}b_q-\lambda-2)}{\Gamma(\text{Re}b_q-1)\Gamma(\text{Re}b_q-\lambda-1)} \\
&= \left(\prod_{m=1}^{q-1} \frac{\text{Re}b_m-1}{\text{Re}b_m-|a_m|-1} \right) \frac{1}{1-|a_q|} \frac{\lambda-1}{\lambda-|a_{q+1}|-1} \frac{\text{Re}b_q-1}{\text{Re}b_q-\lambda-2}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \sum_{n=2}^{\infty} (n+\lambda-1) \left| \frac{\prod_{j=1}^{q+1} (a_j)_{n-1}}{\prod_{i=1}^q (b_i)_{n-1} (1)_{n-1}} a_n \right| \\
&= (A-B)|t| \sum_{n=1}^{\infty} \frac{\lambda(\lambda+1)_n}{(\lambda)_n} \frac{(1)_n}{(2)_n} \left\{ \frac{\prod_{j=1}^{q+1} (|a_j|)_n}{\prod_{i=1}^q (\text{Re}b_i)_n (1)_n} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \lambda(A-B)|t| \left\{ {}_{q+3}F_{q+2} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_{q+1}|, \lambda+1, 1 \\ \text{Reb}_1, \text{Reb}_2, \dots, \text{Reb}_q, \lambda, 2 \end{matrix} ; 1 \right) - 1 \right\} \\
&\leq \lambda,
\end{aligned}$$

by assumption, which completes the proof of Corollary 2.2.

Theorem 2.4 Let a_j ($j = 1, 2, \dots, p$) $\in C \setminus \{0\}$, b_i ($i = 1, 2, \dots, q$) $\in C \setminus \{0\}$, $\text{Re}b_i > 0$ ($i = 1, 2, \dots, q$), and $\sum_{i=1}^q \text{Re}b_i > \sum_{j=1}^p |a_j| + 1$. If $f(z) \in R^t(A, B)$ satisfies

$$(2.10) \quad {}_{p+1}F_{q+1} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, \lambda+1 \\ \text{Re}b_1, \text{Re}b_2, \dots, \text{Re}b_q, \lambda \end{matrix} ; 1 \right) \leq \frac{1}{(A-B)|t|} + 1,$$

then $I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_p}(f) \in C_\lambda$ where $\lambda > 0$.

Proof. Since the proof follows from Lemma 2.3 and by using the method of the proof of Theorem 2.3, we omit the details.

Corollary 2.3 Let a_j ($j = 1, 2, \dots, q+1$) $\in C \setminus \{0\}$, b_i ($i = 1, 2, \dots, q$) $\in C \setminus \{0\}$, $\text{Re}b_m > |a_m| + 1$ ($m = 1, 2, \dots, q-1$), and $\text{Re}b_q - 2 > \lambda > |a_q| + |a_{q+1}|$. If $f(z) \in R^t(A, B)$ satisfies

$$(2.11) \quad \frac{(\text{Re}b_q - 1)\Gamma(\lambda)\Gamma(\lambda - |a_q| - |a_{q+1}|)}{(\text{Re}b_q - \lambda - 2)\Gamma(\lambda - |a_q|)\Gamma(\lambda - |a_{q+1}|)} \\ \times \left(\prod_{m=1}^{q-1} \frac{\text{Re}b_m - 1}{\text{Re}b_m - |a_m| - 1} \right) \leq \frac{1}{(A-B)|t|} + 1,$$

then $I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_{q+1}}(f) \in C_\lambda$ where $\lambda > 0$.

Proof. We note that

$$\begin{aligned}
& {}_{q+2}F_{q+1} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_{q+1}|, \lambda+1 \\ \text{Re}b_1, \text{Re}b_2, \dots, \text{Re}b_q, \lambda \end{matrix} ; 1 \right) \\
&= \left(\prod_{m=1}^{q-1} \frac{\Gamma(\text{Re}b_m)\Gamma(\text{Re}b_m - |a_m| - 1)}{\Gamma(\text{Re}b_m - |a_m|)\Gamma(\text{Re}b_m - 1)} \right) \frac{\Gamma(\text{Re}b_q)\Gamma(\text{Re}b_q - \lambda - 2)}{\Gamma(\text{Re}b_q - 1)\Gamma(\text{Re}b_q - \lambda - 1)} \\
&\quad \times \frac{\Gamma(\lambda)\Gamma(\lambda - |a_q| - |a_{q+1}|)}{\Gamma(\lambda - |a_q|)\Gamma(\lambda - |a_{q+1}|)} \\
&= \left(\prod_{m=1}^{q-1} \frac{\text{Re}b_m - 1}{\text{Re}b_m - |a_m| - 1} \right) \frac{\text{Re}b_q - 1}{\text{Re}b_q - \lambda - 2} \frac{\Gamma(\lambda)\Gamma(\lambda - |a_q| - |a_{q+1}|)}{\Gamma(\lambda - |a_q|)\Gamma(\lambda - |a_{q+1}|)}.
\end{aligned}$$

Hence we observe that

$$\begin{aligned}
& \sum_{n=2}^{\infty} n(n+\lambda-1) \left| \frac{\prod_{j=1}^{q+1} (a_j)_{n-1}}{\prod_{i=1}^q (b_i)_{n-1} (1)_{n-1}} a_n \right| \\
&= (A-B)|t| \sum_{n=1}^{\infty} \frac{\lambda(\lambda+1)_n}{(\lambda)_n} \left\{ \frac{\prod_{j=1}^{q+1} (|a_j|)_n}{\prod_{i=1}^q (\text{Re} b_i)_n (1)_n} \right\} \\
&= \lambda(A-B)|t| \left\{ {}_{q+2}F_{q+1} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_{q+1}|, \lambda+1 \\ \text{Re} b_1, \text{Re} b_2, \dots, \text{Re} b_q, \lambda \end{matrix} ; 1 \right) - 1 \right\} \\
&\leq \lambda,
\end{aligned}$$

by assumption. This completes our proof.

Theorem 2.5 Let a_j ($j = 1, 2, \dots, p$) $\in C \setminus \{0\}$, b_i ($i = 1, 2, \dots, q$) $\in C \setminus \{0\}$, $\text{Re} b_i > 0$ ($i = 1, 2, \dots, q$), and $\sum_{i=1}^q \text{Re} b_i > \sum_{j=1}^p |a_j| + 1$. If

$$\begin{aligned}
(2.12) \quad & k \frac{\prod_{j=1}^p |a_j|}{\prod_{i=1}^q \text{Re} b_i} {}_pF_q \left(\begin{matrix} |a_1| + 1, |a_2| + 1, \dots, |a_p| + 1 \\ \text{Re} b_1 + 1, \text{Re} b_2 + 1, \dots, \text{Re} b_q + 1 \end{matrix} ; 1 \right) \\
& + {}_pF_q \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p| \\ \text{Re} b_1, \text{Re} b_2, \dots, \text{Re} b_q \end{matrix} ; 1 \right) \leq \frac{(A-B)|t|}{1+|B|} + 1,
\end{aligned}$$

then

$$z {}_pF_q \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} ; z^k \right) \in R^t(A, B),$$

where $k \in \mathbb{N}$.

Proof. By Lemma 2.3, it suffices to show that

$$(2.13) \quad T_3 := \sum_{n=2}^{\infty} (1+|B|)(k(n-1)+1) \left| \frac{\prod_{j=1}^p (a_j)_{n-1}}{\prod_{i=1}^q (b_i)_{n-1} (1)_{n-1}} \right| \leq (A-B)|t|.$$

Then we have,

$$\begin{aligned}
T_3 &\leq (1+|B|) \sum_{n=2}^{\infty} (kn - (k-1)) \frac{\prod_{j=1}^p (|a_j|)_{n-1}}{\prod_{i=1}^q (\text{Re} b_i)_{n-1} (1)_{n-1}} \\
&= (1+|B|)k {}_{p+1}F_{q+1} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, 2 \\ \text{Re} b_1, \text{Re} b_2, \dots, \text{Re} b_q, 1 \end{matrix} ; 1 \right)
\end{aligned}$$

$$\begin{aligned}
& -(1 + |B|)(k - 1) {}_pF_q \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p| \\ \text{Reb}_1, \text{Reb}_2, \dots, \text{Reb}_q \end{matrix} ; 1 \right) - (1 + |B|) \\
& = (1 + |B|)k \frac{\prod_{j=1}^p |a_j|}{\prod_{i=1}^q \text{Reb}_i} {}_pF_q \left(\begin{matrix} |a_1| + 1, |a_2| + 1, \dots, |a_p| + 1 \\ \text{Reb}_1 + 1, \text{Reb}_2 + 1, \dots, \text{Reb}_q + 1 \end{matrix} ; 1 \right) \\
& \quad + (1 + |B|) {}_pF_q \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p| \\ \text{Reb}_1, \text{Reb}_2, \dots, \text{Reb}_q \end{matrix} ; 1 \right) - (1 + |B|) \\
& \leq (A - B)|t|
\end{aligned}$$

by (2.12), which completes the proof of Theroem 2.5.

Corollary 2.4 Let a_j ($j = 1, 2, \dots, q + 1$) $\in C \setminus \{0\}$, b_i ($i = 1, 2, \dots, q$) $\in C \setminus \{0\}$, $\text{Re}b_m > |a_m| + 1$ ($m = 1, 2, \dots, q - 1$), and $\text{Re}b_q > |a_q| + |a_{q+1}| + 1$. If

$$\begin{aligned}
(2.14) \quad & \frac{\Gamma(\text{Re}b_q)\Gamma(\text{Re}b_q - |a_q| - |a_{q+1}| - 1)}{\Gamma(\text{Re}b_q - |a_q|)\Gamma(\text{Re}b_q - |a_{q+1}|)} \left(\prod_{m=1}^{q-1} \frac{1}{\text{Re}b_m - |a_m| - 1} \right) \\
& \times \left\{ k \prod_{j=1}^{q+1} |a_j| + \left(\prod_{m=1}^{q-1} \text{Re}b_m - 1 \right) (\text{Re}b_q - |a_q| - |a_{q+1}| - 1) \right\} \\
& \leq \frac{(A - B)|t|}{1 + |B|} + 1,
\end{aligned}$$

then

$$z {}_{q+1}F_q \left(\begin{matrix} a_1, a_2, \dots, a_{q+1} \\ b_1, b_2, \dots, b_q \end{matrix} ; z^k \right) \in R^t(A, B),$$

where $k \in N$.

Proof. We note that

$$\begin{aligned}
& {}_{q+1}F_q \left(\begin{matrix} |a_1| + 1, |a_2| + 1, \dots, |a_{q+1}| + 1 \\ \text{Re}b_1 + 1, \text{Re}b_2 + 1, \dots, \text{Re}b_q + 1 \end{matrix} ; 1 \right) \\
& = \left(\prod_{m=1}^{q-1} \frac{\Gamma(\text{Re}b_m + 1)\Gamma(\text{Re}b_m - |a_m| - 1)}{\Gamma(\text{Re}b_m)\Gamma(\text{Re}b_m - |a_m|)} \right) \frac{\Gamma(\text{Re}b_q + 1)\Gamma(\text{Re}b_q - |a_q| - |a_{q+1}| - 1)}{\Gamma(\text{Re}b_q - |a_q|)\Gamma(\text{Re}b_q - |a_{q+1}|)} \\
& = \left(\prod_{m=1}^{q-1} \frac{\text{Re}b_m}{\text{Re}b_m - |a_m| - 1} \right) \frac{\Gamma(\text{Re}b_q + 1)\Gamma(\text{Re}b_q - |a_q| - |a_{q+1}| - 1)}{\Gamma(\text{Re}b_q - |a_q|)\Gamma(\text{Re}b_q - |a_{q+1}|)}.
\end{aligned}$$

From above equality and (2.6), we have the result of Corollary 2.4.

3. Uniformly starlikeness and convexity

A function $f(z) \in \mathcal{A}$ is said to be uniformly starlike in U if it satisfies

$$(3.1) \quad \operatorname{Re} \left\{ \frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} \right\} \geq 0$$

for all $(z, \zeta) \in U \times U$. We denote by UST the subclass of \mathcal{A} consisting of all uniformly starlike functions in U . Further, a function $f(z) \in \mathcal{A}$ is said to be uniformly convex in U if and only if

$$(3.2) \quad \operatorname{Re} \left\{ 1 + (z - \zeta) \frac{f''(z)}{f'(z)} \right\} \geq 0$$

for all $(z, \zeta) \in U \times U$. We also denote by UCV the class of all such functions.

The classes UST and UCV were defined by Goodman [6,7] and studied recently by Rønning [13]. By the result of Rønning [13], we see that $f(z) \in UCV$ if and only if

$$(3.3) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in U).$$

In view of definitions of UST and UCV , we define the following classes :

Definition 3.1 A function $f(z)$ in \mathcal{A} is said to be a member of the class $UST(\alpha)$ if it satisfies

$$(3.4) \quad \operatorname{Re} \left\{ \frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} \right\} \geq \alpha \quad ((z, \zeta) \in U \times U)$$

for some real α ($0 \leq \alpha < 1$).

Definition 3.2 A function $f(z)$ belonging to \mathcal{A} is called as a member of the class $UCV(\alpha)$ if and only if

$$(3.5) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \alpha \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in U)$$

for some real α ($\alpha \geq 0$).

Note that $UST(\alpha) \subset UST$ ($0 \leq \alpha < 1$), $UCV(\alpha) \subset UCV$ ($\alpha \geq 1$) and $UCV \subset UCV(\alpha)$ ($0 \leq \alpha < 1$). Now, we derive the following lemmas for functions $f(z) \in \mathcal{A}$ to be in the classes $UST(\alpha)$ and $UCV(\alpha)$.

Lemma 3.1 If $f(z) \in \mathcal{A}$ satisfies $\sum_{n=2}^{\infty} n(n(\alpha+1) - \alpha)|a_n| \leq 1$, then $f(z)$ is in $UCV(\alpha)$.

Proof. It suffices to show that

$$(3.6) \quad \alpha \left| \frac{zf''(z)}{f'(z)} \right| - \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) \leq 1.$$

We have

$$\begin{aligned} \alpha \left| \frac{zf''(z)}{f'(z)} \right| - \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) &\leq (\alpha + 1) \left| \frac{zf''(z)}{f'(z)} \right| \\ &= \left| \frac{(\alpha + 1) \sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} na_n z^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} (\alpha + 1)n(n-1)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|}. \end{aligned}$$

Now this last expression is bounded above by 1 provided that $\sum_{n=2}^{\infty} n(n(\alpha + 1) - \alpha)|a_n| \leq 1$.

Lemma 3.2 *If $f(z) \in \mathcal{A}$ satisfies $\sum_{n=2}^{\infty} ((3 - \alpha)n - 2)|a_n| \leq 1 - \alpha$, then $f(z)$ is in $UST(\alpha)$.*

Proof. It suffices to show that

$$(3.7) \quad \left| \frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} - 1 \right| \leq 1 - \alpha.$$

We have

$$\begin{aligned} \left| \frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} - 1 \right| &= \left| \frac{\sum_{n=2}^{\infty} a_n(z^{n-1} + z^{n-2}\zeta + \dots + \zeta^{n-1}) - \sum_{n=2}^{\infty} na_n z^{n-1}}{1 + \sum_{n=2}^{\infty} na_n z^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} 2(n-1)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|}, \end{aligned}$$

which is bounded above by $1 - \alpha$ if $\sum_{n=2}^{\infty} ((3 - \alpha)n - 2)|a_n| \leq 1 - \alpha$.

Theorem 3.1 *Let a_j ($j = 1, 2, \dots, p$) $\in C \setminus \{0\}$, b_i ($i = 1, 2, \dots, q$) $\in C \setminus \{0\}$, $\operatorname{Re} b_i > 0$ ($i = 1, 2, \dots, q$), and $\sum_{i=1}^q \operatorname{Re} b_i > \sum_{j=1}^p |a_j| + 1$. If $f(z) \in R^t(A, B)$ satisfies*

$$(3.8) \quad (1 + \alpha) \frac{\prod_{j=1}^p |a_j|}{\prod_{i=1}^q \operatorname{Re} b_i} {}_pF_q \left(\begin{matrix} |a_1| + 1, |a_2| + 1, \dots, |a_p| + 1 \\ \operatorname{Re} b_1 + 1, \operatorname{Re} b_2 + 1, \dots, \operatorname{Re} b_q + 1 \end{matrix} ; 1 \right) \\ + {}_pF_q \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p| \\ \operatorname{Re} b_1, \operatorname{Re} b_2, \dots, \operatorname{Re} b_q \end{matrix} ; 1 \right) \leq \frac{1}{(A - B)|t|} + 1,$$

then $I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_p}(f) \in UCV(\alpha)$.

Proof. By Lemma 3.1, we need only to show that

$$(3.9) \quad S_1 := \sum_{n=2}^{\infty} n(n(\alpha+1) - \alpha) \left| \frac{\prod_{j=1}^p (a_j)_{n-1}}{\prod_{i=1}^q (b_i)_{n-1} (1)_{n-1}} a_n \right| \leq 1.$$

From Lemma 2.2, we have,

$$\begin{aligned} S_1 &\leq (A-B)|t| \sum_{n=2}^{\infty} (n(\alpha+1) - \alpha) \frac{\prod_{j=1}^p (|a_j|)_{n-1}}{\prod_{i=1}^q (\text{Re} b_i)_{n-1} (1)_{n-1}} \\ &= (A-B)|t|(\alpha+1) {}_{p+1}F_{q+1} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, 2 \\ \text{Re} b_1, \text{Re} b_2, \dots, \text{Re} b_q, 1 \end{matrix} ; 1 \right) \\ &\quad - (A-B)|t|\alpha {}_pF_q \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p| \\ \text{Re} b_1, \text{Re} b_2, \dots, \text{Re} b_q \end{matrix} ; 1 \right) - (A-B)|t| \\ &= (A-B)|t|(\alpha+1) \frac{\prod_{j=1}^p |a_j|}{\prod_{i=1}^q \text{Re} b_i} {}_pF_q \left(\begin{matrix} |a_1|+1, |a_2|+1, \dots, |a_p|+1 \\ \text{Re} b_1+1, \text{Re} b_2+1, \dots, \text{Re} b_q+1 \end{matrix} ; 1 \right) \\ &\quad + (A-B)|t| {}_pF_q \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p| \\ \text{Re} b_1, \text{Re} b_2, \dots, \text{Re} b_q \end{matrix} ; 1 \right) - (A-B)|t| \\ &\leq 1 \end{aligned}$$

by (3.8), which completes the proof of Theorem 3.1.

Corollary 3.1 Let a_j ($j = 1, 2, \dots, q+1$) $\in C \setminus \{0\}$, b_i ($i = 1, 2, \dots, q$) $\in C \setminus \{0\}$, $\text{Re} b_m > |a_m| + 1$ ($m = 1, 2, \dots, q-1$), and $\text{Re} b_q > |a_q| + |a_{q+1}| + 1$. If $f(z) \in R^t(A, B)$ satisfies

$$\begin{aligned} (3.10) \quad &\frac{\Gamma(\text{Re} b_q) \Gamma(\text{Re} b_q - |a_q| - |a_{q+1}| - 1)}{\Gamma(\text{Re} b_q - |a_q|) \Gamma(\text{Re} b_q - |a_{q+1}|)} \left(\prod_{m=1}^{q-1} \frac{1}{\text{Re} b_m - |a_m| - 1} \right) \\ &\quad \times \left\{ (\alpha+1) \prod_{j=1}^{q+1} |a_j| + \left(\prod_{m=1}^{q-1} \text{Re} b_m - 1 \right) (\text{Re} b_q - |a_q| - |a_{q+1}| - 1) \right\} \\ &\leq \frac{1}{(A-B)|t|} + 1, \end{aligned}$$

then $I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_{q+1}}(f) \in UCV(\alpha)$.

Proof. Since the proof is similarly the proof of Corollary 2.4, we omit the details.

Theorem 3.2 Let a_j ($j = 1, 2, \dots, p$) $\in C \setminus \{0\}$, b_i ($i = 1, 2, \dots, q$) $\in C \setminus \{0\}$, $\text{Re} b_i > 0$ ($i = 1, 2, \dots, q$), and $\sum_{i=1}^q \text{Re} b_i > \sum_{j=1}^p |a_j|$. If $f(z) \in R^t(A, B)$ satisfies

$$(3.11) \quad (3 - \alpha) {}_pF_q \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p| \\ \text{Re} b_1, \text{Re} b_2, \dots, \text{Re} b_q \end{matrix} ; 1 \right) - 2 {}_{p+1}F_{q+1} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, 1 \\ \text{Re} b_1, \text{Re} b_2, \dots, \text{Re} b_q, 2 \end{matrix} ; 1 \right) \leq (1 - \alpha) \left(\frac{1}{(A - B)|t|} + 1 \right),$$

then $I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_p}(f) \in UST(\alpha)$, for some α ($0 \leq \alpha < 1$).

Proof. By Lemma 3.2, we need only to show that

$$(3.12) \quad S_2 := \sum_{n=2}^{\infty} ((3 - \alpha)n - 2) \left| \frac{\prod_{j=1}^p (a_j)_{n-1}}{\prod_{i=1}^q (b_i)_{n-1} (1)_{n-1}} a_n \right| \leq 1 - \alpha.$$

From Lemma 2.2, we have,

$$\begin{aligned} S_2 &\leq (A - B)|t|(3 - \alpha) \sum_{n=2}^{\infty} \frac{\prod_{j=1}^p (|a_j|)_{n-1}}{\prod_{i=1}^q (\text{Re} b_i)_{n-1} (1)_{n-1}} \\ &\quad - 2(A - B)|t| \sum_{n=2}^{\infty} \frac{\prod_{j=1}^p (|a_j|)_{n-1}}{\prod_{i=1}^q (\text{Re} b_i)_{n-1} (1)_{n-1}} \frac{(1)_{n-1}}{(2)_{n-1}} \\ &= (A - B)|t|(3 - \alpha) {}_pF_q \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p| \\ \text{Re} b_1, \text{Re} b_2, \dots, \text{Re} b_q \end{matrix} ; 1 \right) \\ &\quad - 2(A - B)|t| {}_{p+1}F_{q+1} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, 1 \\ \text{Re} b_1, \text{Re} b_2, \dots, \text{Re} b_q, 2 \end{matrix} ; 1 \right) - (1 - \alpha)(A - B)|t| \\ &\leq 1 - \alpha \end{aligned}$$

by (3.11), which completes the proof of Theorem 3.2.

Corollary 3.2 Let a_j ($j = 1, 2, \dots, q + 1$) $\in C \setminus \{0\}$, b_i ($i = 1, 2, \dots, q$) $\in C \setminus \{0\}$, $\text{Re} b_m > |a_m| + 1$ ($m = 1, 2, \dots, q - 1$), and $\text{Re} b_q > |a_q| + 1$, $|a_{q+1}| < 1$. If $f(z) \in R^t(A, B)$ satisfies

$$(3.13) \quad \left\{ (3 - \alpha) \frac{\Gamma(\text{Re} b_q - 1) \Gamma(\text{Re} b_q - |a_q| - |a_{q+1}|)}{\Gamma(\text{Re} b_q - |a_q| - 1) \Gamma(\text{Re} b_q - |a_{q+1}|)} - \frac{2}{1 - |a_{q+1}|} \right\} \\ \times \left(\prod_{m=1}^q \frac{\text{Re} b_m - 1}{\text{Re} b_m - |a_m| - 1} \right) \leq (1 - \alpha) \left(\frac{1}{(A - B)|t|} + 1 \right),$$

then $I_{b_1, b_2, \dots, b_q}^{a_1, a_2, \dots, a_{q+1}}(f) \in UST(\alpha)$, for some α ($0 \leq \alpha < 1$).

Proof. We note that

$$\begin{aligned} & {}_{q+2}F_{q+1} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_{q+1}|, 1 \\ \text{Re}b_1, \text{Re}b_2, \dots, \text{Re}b_q, 2 \end{matrix} ; 1 \right) \\ &= \left(\prod_{m=1}^{q-1} \frac{\text{Re}b_m - 1}{\text{Re}b_m - |a_m| - 1} \right) \frac{\text{Re}b_q - 1}{(\text{Re}b_q - |a_q| - 1)(1 - |a_{q+1}|)}. \end{aligned}$$

From above equality and (2.6), we have the result of Corollary 3.2.

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